Traffic in the Network

Allen and Arkolakis (2020)

"The Welfare Effects of Transportation Infrastructure Improvements"
find out more here: https://sites.google.com/site/treballen/research

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WG Endogenous Network

December 2020

Motivation

- **Quantitative Spatial Economics**

  Tractable quantitative models of economic geography

  - Many Locations, General equilibrium and Structural gravity equation
  - Allow to conduct counterfactual analysis

  (Anderson, Van wincoop, 2003; Arkolakis et al. 2012, Allen, Arkolakis, 2014, Head, Mayer, 2014, Ahlfeldt et al. 2015, Redding, Rossi-H., 2017, Heiblich et al. 2020)

- **My current project**

  How mass-transit shapes local labor markets?

  - *RER* in Paris: large investment in efficient public transport
  - *Disentangle* investment, flows and economic outcomes.. modal shift?

- **This paper**

  The welfare effects of transportation infrastructure

  - Transportation **infrastructure** improvements
  - Endogenous traffic **congestion**
  - New approach: **path level** instead of origin-destination dyads

Summary

How transportation infrastructure investment impacts welfare?

- **Economic problem**: utility/profit maximization + space
  - *Location choice* (commuting/trade, housing market)
  - *Market failure*: agglomeration and dispersion forces
- **Routing problem**: to travel from a to b through sequence of paths
  - *Previously*: optimal route (Dijkstra or A* and Fast Marching Method)
  - *New*: allow for **alternative routes** + ripple effect
  - *New*: from routing to traffic
- **Feedback loop**: commute .. traffic .. congestion .. commuting cost ..
  - Define the endogenous **traffic gravity**
  - Endogenous transportation cost: *Infrastructure* and *Congestion*.
  - Spatial equilibrium with traffic congestion
- **Empirical results**
  - Use **traffic data** (geographic and urban)
  - Return of Investment: *large and heterogeneous*

Structural Gravity Equation (SGE)

- **Ingredients** (Urban model) Individual preferences $\rightarrow$ location choice with $\kappa_{i,j}$ the deterministic route cost (least-cost route)

$$u_{i,j}(w) = \max_{C, H_{R_i}, (i,j)} \left(\frac{C}{\beta}\right)^{\beta} \left(\frac{H_{R_i}}{1-\beta}\right)^{1-\beta} \frac{u_i}{\kappa_{i,j}} \nu_{i,j}(\omega)$$

Implies the following indirect utility: $v_{i,j}(w) = \frac{u_i w_j}{\kappa_{i,j} q_{R_i}^{1-\beta}} \nu_{i,j}(\omega)$ and idiosyncratic preference (discrete choice model): $\nu_{i,j}(\omega)$ drawn from a Frechet $G(\nu) = e^{\nu^{-\epsilon}}$, with shape $\epsilon > 1$. Flows, sum over ω (see Eaton, Kortum, 2002; eq. 13 in A.A.2019)

$$\pi_{i,j} = \frac{L_{i,j}}{\overline{L}} = \frac{(u_i w_j)^{\epsilon} (\kappa_{i,j} q_{R_i}^{1-\beta})^{-\epsilon}}{\sum_{k,l} (u_k w_l)^{\epsilon} (\kappa_{k,l} q_{R_k}^{1-\beta})^{-\epsilon}} \quad (1)$$

- **Spatial distribution of economic activities**
  see for instance Tsivanidis, 2020

$$R_i = \sum_l \overline{L} \pi_{i,l} = g \; (u_i q_{R_i}^{\beta-1})^{\epsilon} \; \Pi_i \quad \text{and} \quad L_j = \sum_l \overline{L} \pi_{l,j} = g \; w_j^{\epsilon} \; \Phi_j \quad (2)$$

Commuting market access of residents $\Pi_i = \sum_l (\frac{w_l}{\kappa_{i,l}})^{\epsilon}$ and firms $\Phi_j = \sum_l (\frac{u_l}{\kappa_{l,j}})^{\epsilon}$

$\Rightarrow$ **Deterministic route cost, origin-destination level!**

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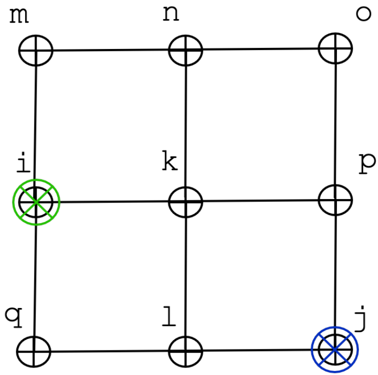
The endogenous routing problem



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- **Route decomposition**: the network is composed of **paths** (edges). To go from i to j a commuter (or a trader) ω choose a **route** $r(\omega)$: *a sequence of path* $\{p_{r_0=i,r_1},..,p_{r_{K-1},r_K=j}\}$ (K the route degree). Individual commuting cost for ω is

$$\tau_{i,j}^{r(\omega)} = \prod_{l=0}^{K(r(\omega))} t_{p_{r_l(\omega),r_{l+1}(\omega)}}$$

decomposes route cost into the sum of path cost $(t_{p_{k,l}})$ allowing for alternatives

- **Gravity equation with alternative routes**
get the gravity equation with endogenous commuting/trade cost

$$v_{i,j}(w) = \frac{u_i w_j}{\prod_{l=0}^{K-1} t_{p_{r_l(\omega),r_{l+1}(\omega)}}} \nu_{i,j,r}^{\omega} \quad \Rightarrow \quad \pi_{i,j} = \frac{L_{i,j}}{L} = \frac{(u_i w_j)^{\epsilon} (\tau_{i,j})^{-\epsilon}}{\sum_{k,l} (u_k w_l)^{\epsilon} (\tau_{k,l})^{-\epsilon}} \tag{3}$$

where the *expected* commuting cost (eq. 4 in A.A.2019) is

$$\tau_{i,j} \equiv \left(\sum_{r \in \mathcal{R}_{i,j}} \left(\tau_{i,j}^{r}\right)^{-\epsilon} \right)^{-\frac{1}{\epsilon}} = \left(\sum_{r \in \mathcal{R}_{i,j}} \left(\prod_{l=0}^{K(r)} t_{p_{r_l,r_{l+1}}} \right)^{-\epsilon} \right)^{-\frac{1}{\epsilon}} \tag{4}$$

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The adjacency matrix approach

- **Network topography**: the weighted adjacency matrix

$$A = [a_{i,j} \equiv t_{i,j}^{-\epsilon}]$$

paths are absent: $a_{i,j} = 0$; cost-less: $a_{i,j} = 1$ (teleport.); or costly: $0 < a_{i,j} < 1$

- **Route cost**: Total alternative route cost by path degree
$\rightarrow$ fix the number of nodes and sum the cost of alternative routes

$$\forall r \in \mathcal{R}_{i,j} | K(r) = 0 : \qquad \tau_{i,j}^{0} = A_{i,i}^{0} = 1 \text{ or } = A_{i,j}^{0} = 0$$

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- **Path integral formulation** of aggregated route cost

$$\tau_{i,j} = \left(\sum_{r \in \mathcal{R}_{i,j}} \left(\tau_{i,j}^{r} \right)^{-\epsilon} \right)^{-\frac{1}{\epsilon}} \Rightarrow \tau_{i,j} = \left(\sum_{K=0}^{\infty} A_{i,j}^{K} \right)^{-\frac{1}{\epsilon}}$$

i any possible path is accounted for (without weight)
ii constant elasticity of substitution correspond to idiosyncratic pref. heterogeneity

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Analytical solution of the Routing problem

- **Leontief inverse** to solve the infinit sum of power matrix (see Bell, 1995)

$$A^0 = \sum_{K=0}^{\infty} \mathrm{A}^K - \sum_{K=1}^{\infty} \mathrm{A}^K = I_N$$

$$\Rightarrow \qquad (I_N - A) \sum_{K=0}^{\infty} \mathrm{A}^K = I_N$$

if A is sparse: $\rho(\mathrm{A}) = \max\{|\lambda_1|, .., |\lambda_N|\} < 1 \Leftrightarrow \lim_{K\to\infty} A^K = 0$

$$\sum_{K=0}^{\infty} \mathrm{A}^K = (\mathrm{I} - \mathrm{A})^{-1} \equiv \mathrm{B} = [b_{i,j}]$$

allows to get an analytical formulation of route cost with all alternatives

$$\tau_{i,j} = b_{i,j}^{-1/\epsilon} \tag{5}$$

mapping from path cost $(t_{k,l})$ to commuting costs $(\tau_{i,j})$.

\+ we can adjust previous gravity equation with a generalized **integral path formulation** for route cost

→ but we would prefer **endogenous** route cost, accounting for

i Infrastructure [exogenous quality]

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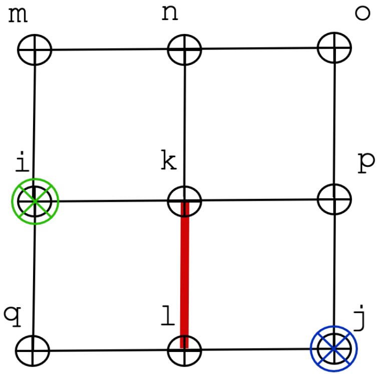
  - ii Congestion [endogenous spillovers]

Path intensity



From routing to traffic

- **Intensity** of path (k, l) use on the way from i to j, corresponds to the elasticity of commuting cost to the specific cost of path (k, l):

$$\pi_{i,j}^{k,l} = \frac{\partial ln(\tau_{i,j})}{\partial ln(t_{k,l})}$$

It is the probability that the path (k, l) to be used on the way from i to j (see Akamatsu, 1996):

$$\pi_{i,j}^{k,l} = \left(\frac{\tau_{i,j}}{\tau_{i,k}\ t_{k,l}\ \tau_{l,j}} \right)^{\epsilon} \quad (6)$$

out of the way paths are used less, as they are more costly to reach, high $\tau_{i,k}\tau_{l,j}$

- **Traffic** sum of all commuters $L_{i,j}$ on the path (k, l) weighted by intensity

$$\Xi_{k,l} = \sum_{i,j} L_{i,j}\ \pi_{i,j}^{k,l} \quad (7)$$

$\Xi_{k,l}$: traffic on the paths (k, l) and $L_{i,j}$: commuting flows (origin/destination)

- **Gravity equation for Traffic** (take: $\Xi_{k,l} = (\frac{1}{t_{k,l}})^{\epsilon} \sum_{i,j} ..$)

Π_k; Φ_l residential and firm commuting market access (the asymmetric multilateral resistance terms) (eq.24 A.A.2020)

$$\Xi_{k,l} = t_{k,l}^{-\epsilon} \Pi_k^{-\epsilon} \Phi_l^{-\epsilon} \quad (8)$$

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From traffic to congestion

- **Infrastructure and Congestion**
  - $\overline{\mathrm{T}} \equiv [\bar{t}_{k,l}]$ is the *infrastructure* matrix (raw quality: road, rail, ..)
  - λ is the strength of traffic *congestion*

$$t_{k,l} = \bar{t}_{k,l} \left(\Xi_{k,l}\right)^{\lambda}$$

Combined with the traffic gravity equation, that is $\Xi_{k,l} = t_{k,l}^{-\epsilon} \Pi_k^{-\epsilon} \Phi_l^{-\epsilon}$ (eq 8)

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- **Road Improvement** (see Duranton and Turner, 2011); improvement $=> \bar{t}_{k,t} \searrow$

$$\frac{\partial ln \Xi_{k,l}}{\partial ln \bar{t}_{k,l}} = -\frac{\epsilon}{1+\epsilon\lambda} \quad (10)$$

unanswered so far: Π and Φ depend on $\frac{\tau_{i,j}}{\tau_{i,k}\tau_{l,j}}$ thus on $\tau_{i,j}$ that depends on $\bar{t}_{k,l}$ .. ?

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Time break: *so far, so good?*

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Location choice + Routing problem

Gravity: $$\pi_{i,j} = \frac{L_{i,j}}{\overline{L}} = \frac{b_{i,j}(u_i w_j)^{\epsilon}}{\overline{W}}$$

Market access: $$\Pi_i = \sum_l b_{i,l} L_l \Phi_l \quad \text{and} \quad \Phi_i = \sum_l b_{l,j} L_l \Pi_l$$

Routing to Traffic

Intensity: $$\pi_{i,j}^{k,l} = \left(\frac{\tau_{i,j}}{\tau_{i,k} t_{k,l} \tau_{l,j}}\right)^{\epsilon}$$

Traffic: $$\Xi_{k,l} = \sum_{i,j} L_{i,j} \pi_{i,j}^{k,l}$$

Infrastructure $\overline{t}$, and Congestion λ

Endogenous costs: $$t_{k,l} = \overline{t}_{k,l}^{\frac{1}{1+\epsilon\lambda}} \Pi_k^{-\frac{\epsilon\lambda}{1+\epsilon\lambda}} \Phi_l^{-\frac{\epsilon\lambda}{1+\epsilon\lambda}}$$

Endogenous traffic: $$\Xi_{k,l} = \overline{t}_{k,l}^{-\frac{\epsilon}{1+\epsilon\lambda}} \Pi_k^{-\frac{\epsilon}{1+\epsilon\lambda}} \Phi_l^{-\frac{\epsilon}{1+\epsilon\lambda}}$$

$\Rightarrow$ **Spatial equilibrium?**

Spatial Equilibrium with Spillovers

- **Market Failures** agglomeration and dispersion forces, which are the *local productivity* v_i and *residential amenities* u_i, with elasticity to jobs, resp. residential population $-1 < \alpha, \beta < 1$

$$v_i = \overline{v}_i L_i^{\alpha} \quad \text{and} \quad u_i = \overline{u}_i R_i^{\beta}$$

- **Equilibrium system** gravity + market closing conditions with $\chi \equiv \frac{L^{\alpha+\beta}}{W}$
  Allen, Arkolakis, 2014; Heiblich et al. 2020

$$(R_i)^{1-\epsilon\beta} = \chi \sum_j \tau_{i,j}^{-\epsilon} \overline{u}_i^{\epsilon} \overline{v}_j^{\epsilon} (L_i)^{\epsilon\alpha} \tag{11}$$

$$(L_i)^{1-\epsilon\alpha} = \chi \sum_j \tau_{j,i}^{-\epsilon} \overline{u}_j^{\epsilon} \overline{v}_i^{\epsilon} (R_i)^{\epsilon\beta} \tag{12}$$

=> link with traffic congestion

$$\tau_{i,j} = b_{i,j}^{-\frac{1}{\epsilon}} = f(t, \epsilon) = f(t(\overline{t}, \Pi, \Phi), \epsilon) = f(t(\overline{t}, L, R), \epsilon)$$

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Spatial Equilibrium with Congestion

- **Spatial distribution of economic activity**

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- **Existence and Uniqueness**
  - More parameters, but same number of endogenous variables .. and equations (2N) .. but **no more linear!**
  - $\overline{\mathrm{T}}$ strongly connected and $\overline{v}_i, \overline{u}_i, R_i, L_I > 0 \Rightarrow$ *existence*
  - in addition, if $\alpha < \frac{1}{2}(\frac{1}{\epsilon} - \lambda)$ and $\beta < \frac{1}{2}(\frac{1}{\epsilon} - \lambda) \Rightarrow$ *uniqueness*
- **Solution?**
  - Solve for the *fixed point* (Matlab: iterate on `fmincom`)
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The model in the data

- **Estimate congestion** $t_{k,l} = \bar{t}_{k,l}\left(\Xi_{k,l}\right)^{\lambda}$ and $\ln(t_{k,l}) = \delta_0 \ln(time_{k,l})$

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- **Parameters**:

  Preferences: $\hat{\epsilon} = 6.83$; Productivity spillovers: $\hat{\alpha} = 0.12$;

  Residential spillovers $\hat{\beta} = -0.1$; *Congestion (IV)*: $\hat{\lambda} = 0.071$

- **Benefit** for every additional lane on path (k, l)

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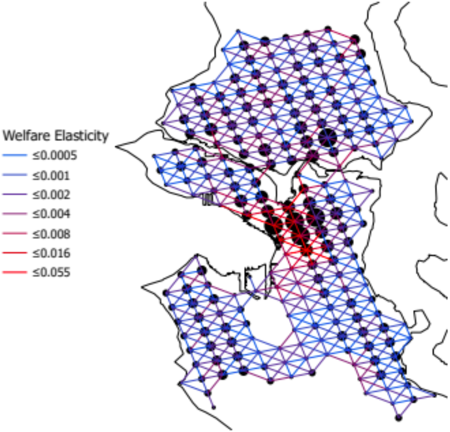
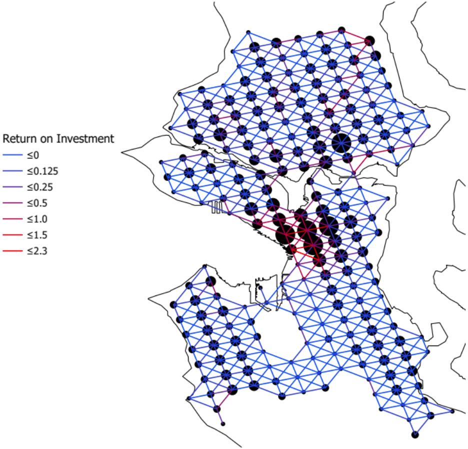
Welfare elasticity: example Seattle



Welfare elasticity to a 1% reduction in transportation costs. (i) Reduction in transportation costs on all links are welfare improving. (ii) The largest welfare elasticities are greatest in the center of the city (downtown). (iii) Welfare elasticities are also higher for the various choke-points in the road network (oftentimes corresponding to bridges).

Return of Investment: example Seattle



Return of investment for a 1% decrease of transportation cost. Improving the average link in Seattle yields an annual return of 16.8% for the residents of the city. There is substantial heterogeneity, with the highest returns are concentrated in the center of the city or between downtown Seattle and another part of the city. Nearly half (331 of 692) links in the Seattle road network would generate negative returns of investment.